Start with a four port matrix

$$General\ 4\ port\ S\ matrix = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix}$$

Applying symmetry $S_{ij} = S_{ji}$

$$General\ Symmetric\ 4\ port\ S\ matrix = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{22} & S_{23} & S_{24} \\ S_{13} & S_{23} & S_{33} & S_{34} \\ S_{14} & S_{24} & S_{34} & S_{44} \end{bmatrix}$$

Applying double symmetry $S_{11} = S_{22} = S_{33} = S_{44}$ and $S_{14} = S_{23}$, $S_{13} = S_{24}$, $S_{12} = S_{34}$ yields:

$$Double\ Symmetric\ 4\ port\ S\ matrix = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{11} & S_{14} & S_{13} \\ S_{13} & S_{14} & S_{11} & S_{12} \\ S_{14} & S_{13} & S_{12} & S_{11} \end{bmatrix}$$

$[S] = [S]^T$ (The transpose of a symmetric S-matrix changes nothing compared to the original matrix.)

The S-matrix is unitary for lossless networks due to conservation of power laws, hence:

$[S]^{T*}[S] = I = [S]^*[S]$ or in its expanded form:

$$\begin{bmatrix} S_{11}^* & S_{12}^* & S_{13}^* & S_{14}^* \\ S_{12}^* & S_{11}^* & S_{14}^* & S_{13}^* \\ S_{13}^* & S_{14}^* & S_{11}^* & S_{12}^* \\ S_{14}^* & S_{13}^* & S_{12}^* & S_{11}^* \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{11} & S_{14} & S_{13} \\ S_{13} & S_{14} & S_{11} & S_{12} \\ S_{14} & S_{13} & S_{12} & S_{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now do the element-by-element matrix multiplication using our doubly symmetric 4 port S-parameter matrix

and set the results equal to the I matrix to produce – 16 equations in four complex unknowns.

The first four equations are the first column of the multiplied matrices and are created by multiplying

row 1 by column 1
row 2 by column 1
row 3 by column 1
row 4 by column 1

which produces:

$$S_{11}^*S_{11} + S_{12}^*S_{12} + S_{13}^*S_{13} + S_{14}^*S_{14} = 1$$

$$S_{12}^*S_{11} + S_{11}^*S_{12} + S_{14}^*S_{13} + S_{13}^*S_{14} = 0$$

$$S_{13}^*S_{11} + S_{14}^*S_{12} + S_{11}^*S_{13} + S_{12}^*S_{14} = 0$$

$$S_{14}^*S_{11} + S_{13}^*S_{12} + S_{12}^*S_{13} + S_{11}^*S_{14} = 0$$

repeating this for the remaining columns yields 3 equation sets that are redundant duplicates of the above set

$$S_{11}^{*}S_{12} + S_{12}^{*}S_{11} + S_{13}^{*}S_{14} + S_{14}^{*}S_{13} = 0$$

$$S_{12}^{*}S_{12} + S_{11}^{*}S_{11} + S_{14}^{*}S_{14} + S_{13}^{*}S_{13} = 1$$

$$S_{13}^{*}S_{12} + S_{14}^{*}S_{11} + S_{11}^{*}S_{14} + S_{12}^{*}S_{13} = 0$$

$$S_{14}^{*}S_{12} + S_{13}^{*}S_{11} + S_{12}^{*}S_{14} + S_{11}^{*}S_{13} = 0$$

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Next we simply our first set of equations:

$S_{11}^{*}S_{11} = |S_{11}|^2$, $S_{12}^{*}S_{12} = |S_{12}|^2$, $S_{13}^{*}S_{13} = |S_{13}|^2$, $S_{14}^{*}S_{14} = |S_{14}|^2$

$S_{12}^{*}S_{11} + S_{11}^{*}S_{12}$ = $S_{11}S_{12}^{*} + (S_{11}S_{12}^{*})^{*}$ = $2 * Real(S_{11}S_{12}^{*})$, ● ● ●

Continuing the simplification, the four equations become:

$$|S_{11}|^2 + |S_{12}|^2 + |S_{13}|^2 + |S_{14}|^2 = 1$$

$$Real(S_{11}S_{12}^{*}) + Real(S_{13}S_{14}^{*}) = 0$$

$$Real(S_{11}S_{13}^{*}) + Real(S_{12}S_{14}^{*}) = 0$$

$$Real(S_{11}S_{14}^{*}) + Real(S_{12}S_{13}^{*}) = 0$$

And we have the final version of the constraints on any lossless, doubly symmetric network S matrix

$$\boldsymbol{|S_{11}|^2 + |S_{12}|^2 + |S_{13}|^2 + |S_{14}|^2 = 1}$$

$$\boldsymbol{|S_{11}||S_{12}|cos(\theta_{11} - \theta_{12}) + |S_{13}||S_{14}|cos(\theta_{13} - \theta_{14}) = 0}$$

$$\boldsymbol{|S_{11}||S_{13}|cos(\theta_{11} - \theta_{13}) + |S_{12}||S_{14}|cos(\theta_{12} - \theta_{14}) = 0}$$

$$\boldsymbol{|S_{11}||S_{14}|cos(\theta_{11} - \theta_{14}) + |S_{12}||S_{13}|cos(\theta_{12} - \theta_{13}) = 0}$$